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'LETTERS' SECTION

On a Quantum-Like Structure of Classical Mechanics

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In a previous paper (Franke & Kálnay, 1969), the authors considered the possibility that at least certain physical systems of classical mechanics have, in addition to the commutative structure provided by the standard algebra, a second, non-commutative structure which is $*$ isomorphic to the C^*-algebra of the observables of quantum mechanics.† It was shown that such new structures can —in principle—be induced by the classical Poisson brackets of the basic variables as well as by its anti-symmetric (Dirac, 1950) and symmetric (Droz-Vincent, 1966), (Franke & Kálnay, 1970) generalizations. The new structure is not always isomorphic to the enveloping algebra of the Lie (or Jordan) algebra corresponding to the classical brackets. The rules for the construction of the new structures, provided they exist, were given in general, but the existence problem was left open. Only in a rather specific relativistic case was existence proved and the structure explicitly constructed and shown to be essentially unique (compare Franke & Kálnay, 1969).

In the present research we looked for such new structure in the classical non-relativistic mechanics of a point particle. *We have proved that it exists* and we have advanced its full explicit construction. The new product can be defined using any two dynamical variables (generalized to complex values) which are analytic functions of the coordinates x_k and the momenta p_k of the particle. The elements of the corresponding algebra are the dynamical variables considered as functions of the time.

Notation: $f*f'$ is the classical variable equal to the new product (denoted $*$ product) of the variables f and f'; ff' is the usual commutative product; f^{*n} is the $*$ product of n factors f; $[f,f'] = f*f' - f'*f$; $\{f,f'\}$ is a Poisson bracket. g is a constant of the theory.

† For related work see, for example, Jordan & Sudarshan (1961), Droz-Vincent (1966), Strocchi (1966), Loinger (1962), Vitale (1968) and the References quoted in this last paper.

Some of our results are (compare Jordan & Sudarshan, 1961):

$$x_1^{\bullet n_1} * x_2^{\bullet n_2} * x_3^{\bullet n_3} = x_1^{n_1} x_2^{n_2} x_3^{n_3} \tag{1}$$

$$p_1^{\bullet n_1} * p_2^{\bullet n_2} * p^{\bullet n_3} = p_1^{n_1} p_2^{n_2} p_3^{n_3} \tag{2}$$

$$x_i * p_j = x_i p_j + (ig/2)\,\delta_{ij}; \qquad i, j = 1, 2, 3 \tag{3}$$

$$[x_i, p_j] = ig\,\delta_{ij} = ig\{x_i, p_j\}; \qquad i, j = 1, 2, 3 \tag{4}$$

$$x_1^{\bullet 2} * p_1^{\bullet 2} = x_1^2 p_1^2 + 2ig x_1 p_1 - (1/2) g^2 \tag{5}$$

$$[x_1^{\bullet 2}, p_1^{\bullet 2}] = 4ig x_1 p_1 = ig\{x_1^2, p_1^2\} \tag{6}$$

$$x_1^{\bullet 3} * p_1^{\bullet 3} = x_1^3 p_1^3 + (9ig/2)\, x_1^2 p_1^2 - (9g^2/2)\, x_1 p_1 - (3i/4) g^3 \tag{7}$$

$$[x_1^{\bullet 3}, p_1^{\bullet 3}] = 9ig x_1^2 p_1^2 - (3ig^3/2) \neq ig\{x_1^3, p_1^3\} \tag{8}$$

The involution $f \to f^+$ is such that $x_i^+ = x_i$, $p_i^+ = p_i$.

The values of the $*$ products of the dynamical variables are consistent with the fact that the algebra whose product is the $*$ product is $*$ isomorphic to the algebra of the quantum operators. The inequality of the commutator and the Poisson bracket in equation (8) is a consequence of this. Indeed, the same happens when replacing in the standard way a classical system by its quantum partner, because it was pointed out (Bergmann & Goldberg, 1955), that the quantization rule $i\hbar\{f, f'\} \to [f, f']$ is right for the basic variables x_i, p_i but not for arbitrary f, f'.

The equations which express the $*$ product in terms of the standard algebra can be inverted. *We note that the $*$ product is a product between the classical variables of the system (differentiable functions of the time), so we are not replacing a classical system by a quantized one. We can prove that classical mechanics can be described in two equivalent ways:* The standard one in which the product is ff', and the new one in which the product is $f * f'$. The existence of a complete solution of our program means that [at least for (1) the non-relativistic point particle without spin, and (2) for one relativistic system with spin included (Franke & Kálnay, 1969)] *the difference between Classical and Quantum Mechanics is more a difference of the interpretative rules than of the mathematical structures to be used.*

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